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# A block theoretic proof of Thompson's $A \times B$-lemma

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**Abstract.** We show that Thompson's $A \times B$-lemma can be obtained as a consequence of Brauer's third main theorem.

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Let $k$ be an algebraically closed field of prime characteristic $p$. Brauer's third main theorem [2, Theorem 3], rephrased using Brauer pairs (cf. [1, Theorem 3.13] or [7, Theorem 6.3.14], or also [5, Theorem 7] for a different proof), states that if $b$ is the principal block idempotent of a finite group algebra $kG$, then $\mathrm{Br}_Q(b)$ is the principal block idempotent of $kC_G(Q)$ for any $p$-subgroup $Q$ of $G$. Here $\mathrm{Br}_Q : (kG)^Q \to kC_G(Q)$ denotes the Brauer homomorphism (cf. [3, §1.2] or [6, Theorem 5.4.1]). In particular, if $kG$ has a unique block, then $kC_G(Q)$ has a unique block. Moreover, if $kG$ has a unique block, then $O_{p'}(G) = 1$ because otherwise $\frac{1}{|O_{p'}(G)|} \sum_{g \in O_{p'}(G)} g$ would be a central idempotent in $kG$ different from 1. Also, it is well-known that if $G$ has a self-centralising normal $p$-subgroup, then $kG$ has a unique block (cf. [7, Corollary 6.2.8]). We use these facts to give a proof of the following result.

**Theorem 1** (Thompson's $A \times B$-lemma; cf. [4, Chapter 5, Theorem 3.4]). *Let $A \times B$ be a subgroup of the automorphism group of a finite $p$-group $P$, with $A$ a $p'$-group and $B$ a $p$-group. If $A$ acts trivially on $C_P(B)$, then $A = 1$.*

*Proof.* Consider the group $G = P \rtimes (A \times B)$, where the notation is as in the statement, and suppose that $A$ acts trivially on $C_P(B)$. Note that $S = P \rtimes B$ is the unique Sylow $p$-subgroup of $G$, and that $A$ can be regarded as a $p'$-subgroup of the automorphism group of $S$. Thus $S$ is self-centralising and normal in $G$, and hence $kG$ has a unique block.

By the assumptions, $A$ acts trivially on the group $Q = C_P(B) \times B$. That is, we have $A \leq C_G(Q)$, and hence $C_G(Q) = C_S(Q) \rtimes A$. Since $kG$ has a unique block, $kC_G(Q)$ has a unique block. We now show that $Q$ is self-centralising in

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$S$. Let $x \in C_S(Q)$. Write $x = yu$ for some $y \in P$ and $u \in B$. Since $u \in B$, it follows that conjugation by $u$ preserves the decomposition $Q = C_P(B) \times B$. Thus conjugation by $y$ preserves this decomposition as well. In particular, $y$ normalises $B$. By elementary group theory, it follows that $y$ centralises $B$. Indeed, if $u \in B$, then $yuy^{-1}u^{-1} \in P \cap B = 1$. This shows that $C_S(Q) \leq Q$. Since $Q \leq C_G(A)$, it follows that $C_G(Q) = C_S(Q) \times A$. By the above, $kC_G(Q)$ has a single block, and hence $A = O_{p'}(C_G(Q)) = 1$. □

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